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OPTIMIZATION OF FIBER REINFORCED COMPOSITE STRUCTURES

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Abstract—This paper presents an efficient optimization method, based on strain energy distribution and a numerical search, for the minimum weight design of structures made from fiber reinforced composite materials. The optimum design procedure takes into consideration multiple loading conditions and displacement constraints on the structure. Sample problems consisting of both isotropic and composite elements are solved and the results presented.

1. INTRODUCTION

IN RECENT years considerable attention has been focused on the use of fiber reinforced composites as structural materials in aerospace applications due to their high strength to weight ratio. Additional weight savings may be obtained by efficient design procedures, and the objective of this paper is to describe such a procedure.

Kicher and Chao [1] developed a method based on nonlinear programming for designing stiffened, fiber reinforced composite cylindrical shells. The shells are designed to satisfy the limiting stress and instability constraints. The design variables are reduced to a small number in order to make the problem amenable to solution by mathematical programming techniques. Waddoups *et al.* [2] have used nonlinear programming methods to design a wing box, where the total number of design variables is equal to eighteen. Cairo and Hadcock [3], and Cairo [4] have presented a procedure to select an optimum layup of preselected orientations of boron epoxy laminates for a single element only using the steepest descent method. This approach in conjunction with the fully stressed design procedure has been used by Lansing *et al.* [5] to design wing and empennage structures. The single optimum fiber orientation for an element can be found by the method given by Sandhu [6].

An optimization method, which is based on strain energy distribution and a numerical search based on constraint gradients, is used to design isotropic structures in [7, 8]. The design procedure is explained in the context of the displacement method of finite element analysis. The present paper extends the combined approach to the optimum design of fiber reinforced composite structures. This design procedure takes into account such practical considerations as:

(1) Multiple loading conditions,
(2) A strength criterion,
(3) Displacement constraints,
(4) Minimum number of layers and minimum size requirements.

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In an optimization scheme the design variables are, in general, functions of the geometric properties and the material constants of the elements. In structures made from an isotropic material, the material constants do not change during the design, and thus, the design variables may be considered as functions of the geometric properties only. However, this is not the case in composite structures because both the number of layers and the fiber orientation in each layer, which define the material constants, can change during the design. In an ideal optimization situation there should be freedom to change the fiber orientation in each layer from point to point in the structure. However, manufacturing difficulties preclude such freedom and the general practice is to select a finite number of predetermined fiber directions and to change the number of laminae in each of the directions to obtain an efficient structure. The directions 0°, 90°, +45° and −45° are generally chosen as basic fiber orientations.

2. BASIS FOR RESIZING OF THE ELEMENTS

2.1 *Optimality criterion based on strain energy distribution*

The optimality criterion derived in [7, 8] when modified for composite structures can be stated as follows: "The optimum structure is the one in which the average strain energy density is the same for all layers and for all elements." The proof of the optimality criterion, as given in [10], is valid for composite structures under the following conditions:

(1) The structure is subjected to a single loading condition.
(2) All the elements of the structure have the same density.
(3) There are no displacement constraints on the structure.
(4) There is no limit on the number of laminae required for any element.

The subsequent sections will show how the modified design criterion can be used to obtain a recurrence relation which constructs a path to the optimum number of laminae for a given set of fiber orientations.

2.2 *Recurrence relation based on the optimality criterion*

In the design of an indeterminate system the task of obtaining an optimum design does not end with the definition of the optimality criterion. Thus an effective iterative algorithm for achieving the optimality condition is derived in this section. In order to make provision for the possibility of varying the stress limits per element, the optimality criterion is modified in the following manner: "The optimum design is the one in which the strain energy of each layer bears a constant ratio to its energy capacity." The energy capacity of the fiber reinforced layer is the maximum energy stored in the layer, if the layer is subjected to a limiting stress in the direction of the fiber orientation. This definition of energy capacity is independent of the actual state of stress in the layer; it depends only on the volume of the layer and the magnitude of the limiting stress. The limiting stresses will be different for composite and isotropic elements. If the assumptions made in the last section are satisfied, then the optimality criterion defined in this section differs from the one stated in the last section by a scaling factor only.

The volume of the kth layer in the ith element may be written as

$$V_{i,k} = (\alpha_{i,k}\Lambda)l_i \tag{1}$$

where Λ is the scaling parameter and is the same for all layers and all the elements in a structure. $\alpha_{i,k}$ is the relative design variable. The actual design variable is $\Lambda\alpha_{i,k}$, which represents the thickness of the kth layer of the ith element. $\Lambda\alpha_{i,1}$ for isotropic elements defines the thickness (plate) or cross-sectional area (bar). In equation (1) l_1 represents the area of a plate element or the length of a bar element. The relation between the displacement vector $\mathbf{v}$ and the applied force vector $\mathbf{s}$ may be written as

$$\mathbf{s} = [K]\mathbf{v} \tag{2}$$

where $[K]$ is the stiffness matrix for the complete structure. Introducing the scaling parameter Λ in equation (2) gives

$$\mathbf{s} = \Lambda[K']\mathbf{v} \tag{3}$$

where $[K']$ is the stiffness matrix for the total structure obtained by using the relative design variable $\alpha_{i,k}$. From equation (3) it is seen that the relative displacement vector $\mathbf{v}'$ is related to the actual displacements by the relation

$$\mathbf{v} = \frac{1}{\Lambda}\mathbf{v}'. \tag{4}$$

Similarly, the relative strains and stresses are related to their actual quantities by

$$\boldsymbol{\varepsilon}_i = \frac{1}{\Lambda}\boldsymbol{\varepsilon}'_i \tag{5}$$

and

$$\boldsymbol{\sigma}_{i,k} = \frac{1}{\Lambda}\boldsymbol{\sigma}'_{i,k} \tag{6}$$

where $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\varepsilon}'_i$ are the actual and relative strains in the ith composite element and $\boldsymbol{\sigma}_{i,k}$ and $\boldsymbol{\sigma}'_{i,k}$ are the actual and relative stresses in the kth layer of the ith element. The strain energy capacity of the kth layer in the ith element is given by

$$\tau_{i,k} = \tfrac{1}{2}\tilde{\sigma}_{i,k}\tilde{\varepsilon}_i V_{i,k} \tag{7}$$

where $\tilde{\sigma}_{i,k}$ and $\tilde{\varepsilon}_i$ are the limiting stress and strain, respectively, in the fiber direction. Using equation (1), the strain energy capacity of the kth layer may be written as

$$\tau_{i,k} = \Lambda\tau'_{i,k} \tag{8}$$

where

$$\tau'_{i,k} = \tfrac{1}{2}\tilde{\sigma}_{i,k}\tilde{\varepsilon}_i(\alpha_{i,k}l_i). \tag{9}$$

The strain energy in the kth layer of the ith element is given by

$$u_{i,k} = \tfrac{1}{2}\boldsymbol{\sigma}^t_{i,k}\boldsymbol{\varepsilon}_i(\alpha_{i,k}\Lambda)l_i. \tag{10}$$

Substituting equations (5) and (6) into equation (10) gives

$$u_{i,k} = \frac{1}{\Lambda}u'_{i,k} \tag{11}$$

where

$$u'_{i,k} = \tfrac{1}{2}\boldsymbol{\sigma}'^{t}_{i,k}\boldsymbol{\varepsilon}'_{i}(\alpha_{i,k}l_{i}). \qquad (12)$$

The optimality criterion stated at the beginning of this section and equations (8) and (11) give

$$\Lambda^2 = C^2 u'_{i,k}/\tau'_{i,k} \qquad (13)$$

where C^2 is the constant of proportionality. Multiplying both sides of the above equation by $\alpha^2_{i,k}$ and taking the square root leads to the following equation

$$\alpha_{i,k}\Lambda = C\alpha_{i,k}\sqrt{(u'_{i,k}/\tau'_{i,k})}. \qquad (14)$$

Equation (14) suggests the following recurrence relation for determining the design variable in each iteration cycle

$$(\alpha_{i,k}\Lambda)_{v+1} = C(\alpha_{i,k})_v\sqrt{(u'_{i,k}/\tau'_{i,k})_v} \qquad (15)$$

where v is the iteration number. When the structure is subjected to more than one loading condition, the recurrence relation given by equation (15) is modified by replacing $u'_{i,k}$ by $u'_{i,k(\mathrm{max})}$, which is the measure of the maximum energy over all loading conditions in the kth layer of the ith element.

The mechanics and the use of this iterative algorithm can be explained with the aid of a two variable design space. In Fig. 1, X_1 and X_2 are the two design variables. The line C–C is the boundary between the feasible and nonfeasible regions and is referred to as the constraint surface. The straight lines W–W represent the constant weight planes. If P is the point that represents the optimum design and A_1 an arbitrary point (or a starting design), then the function of the optimization algorithm is to construct an efficient path from A_1 to P. Such a path can be constructed with the aid of the recurrence relation based on the optimality criterion and the scaling procedure discussed later in this section.

A line joining any point in the X_1–X_2 space with the origin, say OA_1, will be called a design line. In the case of bars and membrane plates all the points on a design line have a common relative design vector. In such a case movement on the design line simply

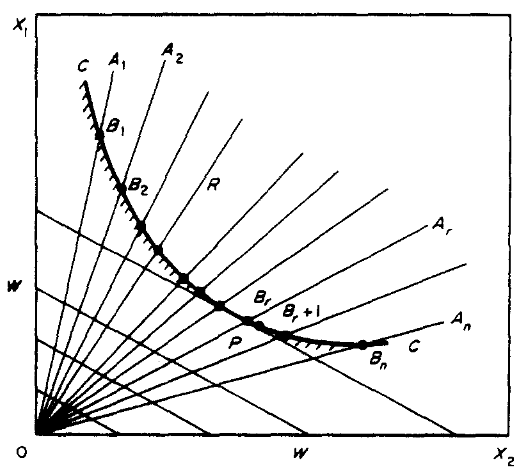

FIG. 1. Design lines in two variable space.

involves manipulation of the scaling parameter Λ. This means one analysis per design line will give all the stress and displacement information anywhere on that line. This movement on a design line is called scaling.

Iteration starts with an arbitrary relative design vector which represents line OA_1. The structure is analyzed with the relative design vector and the point B_1 on the constraint surface is located by adjusting the scaling parameter Λ. B_1 represents the lowest feasible design on OA_1. The weight of the design corresponding to the point B_1 is determined, and it can be used to evaluate the past and the future designs. Then a new design line OA_2 is generated by changing each design variable by equation (15) and normalizing the resulting design vector by the largest of the variables. The structure is reanalyzed with the new relative design vector and the point B_2 on the constraint surface is located by the scaling procedure. The design procedure is terminated after a specified number of iterations or when the difference between two consecutive iterations is less than a stipulated percentage. When the design conditions include multiple loading conditions, constraints on sizes and displacements, etc., the optimum design and the design that satisfied the optimality criterion are not synonymous. In such a case our interest is the optimum design and not necessarily the design satisfying the optimality criterion, and the iterative algorithm based on the optimality criterion should be used only as long as it improves the design. The optimum design usually lies on the path to the design satisfying the optimality criterion. When displacement constraints are present, the range of iteration can be extended by the iterative algorithm based on constraint gradients which is derived in the next Section.

2.3 *Recurrence relation based on constraint gradients*

A search procedure to optimize a structure subjected to displacement constraints was proposed in [7, 8]. A summary of the equations involved in the search is given in this section, and a detailed discussion may be found in [8].

When the structure is subjected to stress constraints only, the recurrence relation, equation (15), based on the strain energy criterion is sufficient to arrive at the optimum design. However, in case the structure is subjected to displacement constraints, the design can be further improved in certain cases by using the algorithm in this section.

The algorithm has the form

$$\boldsymbol{\alpha}_{v+1} = \boldsymbol{\alpha}_v + \beta \mathbf{D}_v \tag{16}$$

where v is the iteration number and β is the step size which determines the rate of approach to the optimum design. β is assumed to be unity for the normal step size. If β is less than unity the approach will be slow, but there will be less chance of missing the optimum between the steps. $\boldsymbol{\alpha}$ in equation (16) is the normalized design vector. The vector **D** is a measure of the difference between the two sets of variables in consecutive iterations. The displacements that are greater than their limiting values are called the active displacements, and they are allowed to exceed their limits by about 20 per cent. The active displacements are then brought to their limiting values by increasing the sizes of the elements in proportion to their influence on these displacements. Only those elements for which an increase in their size reduces the active displacements are allowed to participate in equation (16).

The elements of the vector **D** in equation (16) are given by

$$D_j = B(\mathrm{d}v_m)_j / l_j \tag{17}$$

where B is the constant of proportionality, and $(dv_m)_j$ is the change in the active displacement v_m due to a change in the jth design variable. The magnitude of $(dv_m)_j$ is obtained from the following relation

$$[K]\mathbf{dv}_j = -[\Delta K]_j\mathbf{v} \tag{18}$$

where $[\Delta K]_j$ is the change in the total stiffness matrix due to a change in the size of the jth variable. The value of B is obtained from the condition that the displacement v_m should be brought to its limiting value by increasing the sizes of the elements according to (equation 17).

The magnitude of B is thus given as

$$B = \frac{\Psi_m}{\sum_{p=1}^{n} (dv_p)_j^2/(\alpha_p l_p)} \tag{19}$$

where Ψ_m is the amount by which the displacements are exceeded at the mth constraint. The summation in equation (19) is carried out only over those elements which participate in equation (16).

3. STRENGTH CRITERION FOR THE COMPOSITE ELEMENT

A structural element in a general state of stress requires a strength criterion to determine its ability to withstand the stresses resulting from the applied loads. This strength criterion is one of the factors that determines the location of the constraint surface in Fig. 1. For an isotropic element in a general state of stress the criterion based on energy of distortion (or von Mises criterion) is used for determining the effective strength constraint. A similar criterion by Hill for orthotropic material, modified by Tsai [9], reads as follows:

$$\left(\frac{\bar{\sigma}_x}{F_x}\right)^2 - \frac{\bar{\sigma}_x\bar{\sigma}_y}{F_xF_x} + \left(\frac{\bar{\sigma}_y}{F_y}\right)^2 + \left(\frac{\bar{\sigma}_{xy}}{F_{xy}}\right)^2 \leq 1 \tag{20}$$

where $\bar{\sigma}_x$ and $\bar{\sigma}_y$ are the stresses in the fiber direction and normal to it, respectively; $\bar{\sigma}_{xy}$ is the shear stress, F_x and F_y are the axial and the transverse strengths of the unidirectional composite, and F_{xy} is the shear strength. F_x and F_y have different magnitudes depending on whether σ_x and σ_y are tensile or compressive stresses. It may be pointed out at this stage that even though the criterion given by equation (20) is employed in this paper, a different strength criterion can be incorporated into the computer program with little effort.

4. DESCRIPTION OF THE FINITE ÈLEMENT COMPUTER PROGRAM

The program uses the displacement method to predict the response of the structure to the applied loads. Four types of elements are included in the program: (1) constant strain triangles, (2) quadrilaterals constructed from four constant strain triangles, (3) shear panels, and (4) bars. The triangles and quadrilaterals may be layered composite elements with fiber orientations in the 0°, 90°, +45° and −45° directions. The layers in the +45° and −45° directions are kept equal by taking the average of the strain energies stored in both layers. The direction of the 0° orientation fibers is specified for each composite element in the global coordinate system.

The elements of the structure are resized by the algorithm based on the strain energy criterion, equation (15), and the iteration is continued as long as the design improves. When the design conditions include constraints on displacements, the numerical search algorithm is then activated. The basic steps of the computer program are summarized here:

(1) The design starts with an arbitrary relative design vector.
(2) The structure is analyzed with the relative design vector.
(3) The scaling parameter Λ is evaluated by satisfying the strength criterion and the displacement constraints if they are present. This step represents the scaling procedure to reach the constraint surface (Fig. 1).
(4) The elements of the structure are resized first by the algorithm based on the strain energy criterion and then by the numerical search if constraints on displacements are present.

The steps 2–4 are repeated until either the specified number of cycles are completed or the weight obtained by two consecutive cycles is less than a specified percentage.

5. NUMERICAL EXAMPLES

A few sample problems, both simple and complex, are presented in this section to illustrate the applicability of the theory. The problems consist of members made of aluminum and/or boron-epoxy layered in the 0°, 90°, +45° and −45° directions. The elastic constants for these materials may be found in Table 1. The initial or starting design for boron–epoxy members consists of equal percentages for each orientation. Complete details and intermediate steps for all sample problems may be found in [10].

5.1 *Square plate under single loads*

A square plate was designed for two loading cases. In the first case the plate was subjected to uniaxial tension in the 0° fiber orientation direction. In the second case it was subjected to pure shear. For the case of uniaxial tension, the thickness of the initial design

TABLE 1. MATERIAL PROPERTIES AND ALLOWABLE STRENGTHS

Material	Aluminum	Boron–Epoxy
Elastic modulus E_{11}	$10{\cdot}5 \times 10^6$ lb/in^2	$30{\cdot}0 \times 10^6$ lb/in^2
Elastic modulus E_{22}	$10{\cdot}5 \times 10^6$ lb/in^2	$2{\cdot}7 \times 10^6$ lb/in^2
Poisson's ratio	0·3	0·21
Shear modulus	$4{\cdot}038 \times 10^6$ lb/in^2	$0{\cdot}65 \times 10^6$ lb/in^2
Specific weight	0·1 lb/in^3	0·0725 lb/in^3
Lamina thickness	—	0·005 in
	Allowable strength (kips/in^2)	
$F_x(\tilde{\sigma}_x > 0)$	67·0	176·0
$F_x(\tilde{\sigma}_x > 0)$	67·0	390·0
$F_y(\tilde{\sigma}_y > 0)$	67·0	11·4
$F_y(\tilde{\sigma}_y < 0)$	67·0	44·6
F_{xy}	$F_x/\sqrt{(3{\cdot}0)}$	2·1

(equal percentages) was reduced 61·2 per cent and all fibers became oriented in the 0° direction. For the case of pure shear only, the initial thickness with equal percentages was reduced 46·0 per cent and the fibers became equally oriented in the +45° and −45° directions.

5.2 *Square plate under multiple loads*

A square plate was designed for the four independent loading conditions given in Table 2, each condition consisting of biaxial loading and shear. For this problem a comparison was made with "OPTLAM" [3], a program that uses the method of steepest descent. The strength criterion of the present method was changed to correspond to that employed in "OPTLAM". This criterion is given in Appendix I.

TABLE 2. LOADS ON SQUARE PLATE

Loading condition	N_x (kips/in.)	N_y (kips/in.)	N_{xy} (kips/in.)
1	2·1	−0·6	7·3
2	6·4	8·3	1·85
3	7·3	7·9	19·9
4	1·7	−0·4	6·9

As shown in Table 3, the thickness of the resulting optimum structure that could withstand any of the loading conditions was reduced 21·3 per cent from the initial thickness. The optimum thickness is increased such that the final design contains an integer number of laminae. This increase in thickness allows a number of combinations of laminae, as shown in Table 3, to satisfy all of the design conditions. This result was obtained in the second iteration. As noted from Table 3 both methods give the same results.

5.3 *Rectangular plate with a hole*

A 6 in. × 12 in. plate with a two inch diameter hole at the center is subjected to a tensile load of 15 kips/in. parallel to the 12 in. side. Because of the symmetry only one quarter of

TABLE 3. FIBER ORIENTATIONS FOR PLATE UNDER MULTIPLE LOADS

Design	Number of laminae			Total Number of laminae	Thickness	H (Strength factor)
	0°	90°	+45° −45°			
Initial	28·49	28·49	28·49	113·96	0·5813	1·000
Optimum	11·41	11·89	34·00	91·30	0·4656	1·000
Final	12	12	34	92	0·4692	0·963
OPTLAM'	10	12	35	92	0·4692	0·980
Some other possible designs	11	11	35	92	0·4692	0·981
	12	10	35	92	0·4692	0·985
	11	13	34	92	0·4692	0·988
	13	11	34	92	0·4692	0·992
	14	10	34	92	0·4692	0·998

the plate is considered for the finite element model. The quarter plate idealization has 142 nodes and 138 plate elements (membrane quadrilaterals and triangles). The 0° fibers are in the direction of the load. The weight of the initial design (equal percentages) required to satisfy the strength criterion is 4·84 lb, and is reduced to 0·836 lb in eight iterations. The contours of the number of laminae in each direction are given in Fig. 2.

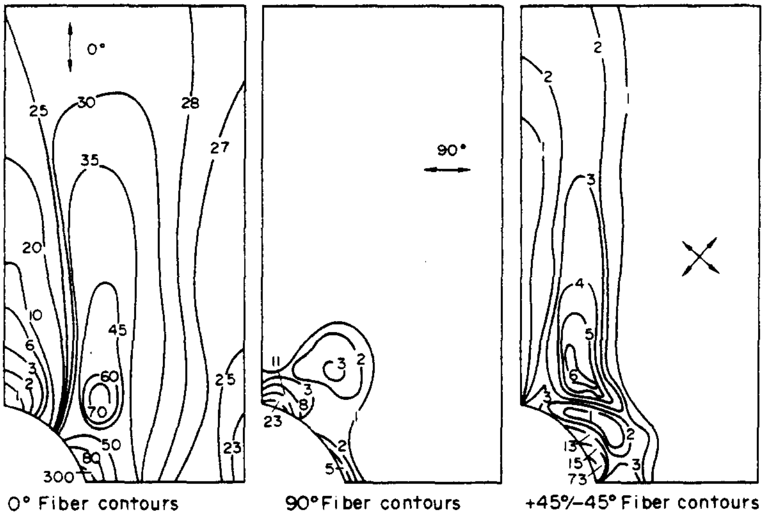

FIG. 2. Fiber orientation contours for a plate with a hole.

5.4 *Cantilever wing*

The planform of the four spar wing idealized with 88 nodes and 170 members is shown in Fig. 3. The spars and ribs are idealized by shear panels and bars, and the top and bottom skins are idealized by membrane quadrilateral elements. The structure is constrained in

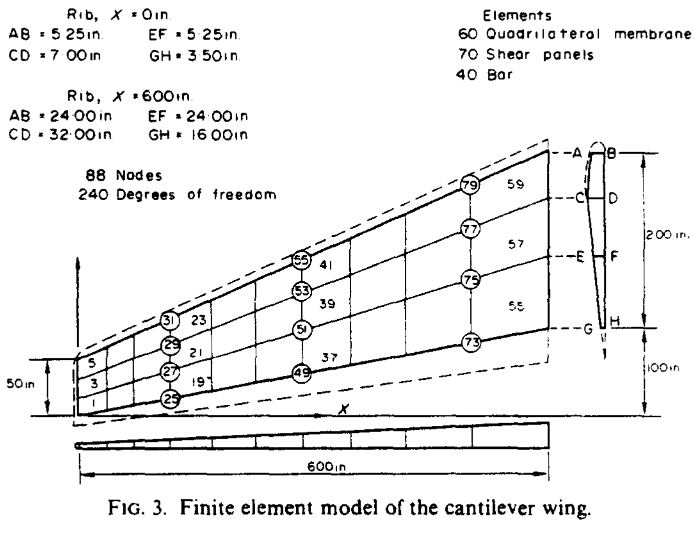

FIG. 3. Finite element model of the cantilever wing.

all directions at the root section. The wing is designed with and without displacement constraints for the two independent loading conditions given in Table 4. The wing is optimized first with all aluminum elements. The design is started with the relative sizes of 1·0 for bar areas and 0·1 for shear and membrane element thicknesses. The wing is next optimized with composite elements replacing the aluminum skins. The 0° fiber orientation is in the direction of the span, and the initial design has equal percentage of fibers in the four fiber directions. The results of these problems are given in Table 5. For the design without displacement constraints, the number of laminae in each fiber direction is given in Fig. 4.

The average CP time (CDC 6600) required for one iteration using the energy criterion is 14·2 seconds for the isotropic and 20·1 for the composite wing. The weights for the displacement constrained design at the end of ten iterations using the energy criterion only are found to be 11,100·7 lb and 3235·6 lb for the isotropic and composite wing,

TABLE 4. DESIGN LOADS ON CANTILEVER WING

Node	Loading Condition I Direction		Loading Condition II Direction	
	Y	*Z*	*Y*	*Z*
2	−2,300	1,800	−2,300	3,000
4		3,600		6,000
6		4,200		2,250
8		5,400		750
10	−2,600	2,000	−2,600	3,400
12		4,100		6,800
14		4,800		2,550
16		6,100		850
18	−2,800	2,300	−2,800	3,800
20		4,600		7,600
22		5,300		2,850
24		6,800		950
26	−3,300	2,600	−3,300	4,400
28		5,300		8,800
30		6,200		3,300
32		7,900		1,100
34	−3,800	3,000	−3,800	5,000
36		6,000		10,000
38		7,000		3,750
40		9,000		1,250
42	−4,200	3,400	−4,200	5,600
44		6,700		11,200
46		7,800		4,200
48		10,100		1,400
50	−4,600	3,700	−4,600	6,200
52		7,400		12,400
54		8,700		4,650
56		11,200		1,550
58	−4,800	3,800	−4,800	6,400
60		7,700		12,800
62		9,000		4,800
64		11,500		1,600

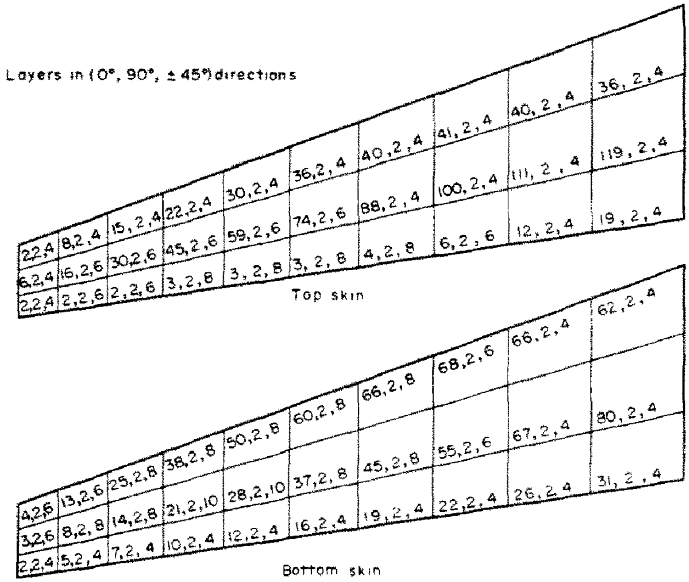

FIG. 4. Optimum design of the cantilever wing.

TABLE 5. WEIGHTS AND DEFLECTIONS OF THE CANTILEVER WING

Isotropic			Composite		
Initial weight	Final weight	Maximum deflection	Initial weight	Final weight	Maximum deflection
5,982·6 lb	4,905·5 lb	80·02 in	6,102·1 lb	2,596·0 lb	43·28 in
16,299·9 lb	10,410·0 lb	36·00 in Imposed	12,046·3 lb	3,091·1 lb	36·00 in Imposed

respectively. The weights given in Table 5 for the design are arrived at after a few numerical search cycles which take an average time of 194 CP seconds per cycle.

6. CONCLUSIONS

It has been demonstrated in this paper that a combined approach based on an optimality criterion and a numerical search procedure can be used successfully to optimize practical structures with composite materials. The recurrence relation based on the optimality criterion can be used for individual composite panels, or it can be incorporated into a finite element approach to optimize built up structures that are commonly encountered in aerospace and other engineering applications. This feature also permits the optimization of each element individually during the optimization of the overall structure. Structures are also designed to satisfy the strength criterion as well as any deflection constraints. Furthermore, the method described here is attractive from the point of computational efficiency and computer time required to design large structures.

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APPENDIX I

Problem 5.2, a square plate under multiple loads, was solved using the strength criterion employed in the program "OPTLAM". This criterion, given here for reference, is written as

$$H = \left(\frac{N_x}{S_x}\right)^2 + \left(\frac{N_y}{S_y}\right)^2 - \left(\frac{N_x N_y}{S_x S_y}\right) + \left(\frac{N_{xy}}{S_{xy}}\right)^2 \leq 1 \tag{21}$$

where N_x, N_y, N_{xy} are the applied forces to the laminae. S_x, S_y and S_{xy} are the allowable strengths of the plate and are given by

$$S_x = (S_1 L + G_2 m)t \qquad N_x \leq 0$$

$$S_x = S_2 S_4 L t \qquad N_x > 0$$

$$S_y = (S_1 M + G_2 l)t \qquad N_y \leq 0$$

$$S_y = S_2 S_4 M t \qquad N_y > 0$$

$$S_{xy} = (nS_3 + (l+m)G_1)t$$

where

$$L = l + n/4 - (n/4)^2/(m + n/4)$$

$$M = m + n/4 - (n/4)^2/(l + n/4) \qquad t = 0{\cdot}0051 \text{ in.}$$

$$S_1 = 230{\cdot}0 \text{ kips/in}^2 \qquad S_2 = 198{\cdot}0 \text{ kips/in}^2$$

$$S_3 = 61{\cdot}2 \text{ kips/in}^2 \qquad S_4 = 0{\cdot}9$$

$$G_1 = 9{\cdot}0 \text{ kips/in}^2 \qquad G_2 = 24{\cdot}0 \text{ kips/in}^2$$

l, m and n are the number of laminae in the 0°, 90° and ±45° directions, respectively.

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Абстракт—В целью расчета конструкций, изготовленных из усиленных волокнами составных материалов, работа дает полезный метод оптимизации, основан на распределению энергии деформации и численном поиске. Методика оптимального расчета учитывает многократное условия нагрузки и ограничения перемещений конструкции. Решаются образцовые задачи, состоящие из изотропных и составных элементов и представляются результаты.